

Graded Clifford algebras of prime global dimension with an action of H_p

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Abstract

In this article we study graded Clifford algebras with a gradation preserving action of automorphisms given by H_p , the Heisenberg group of order p^3 with p prime. After reviewing results in dimensions 3 and 4, we will determine the graded Clifford algebras that are *AS*-regular algebras of global dimension 5 and generalise certain results to arbitrary dimension p .

1 Introduction

In Artin and Schelter [1987], Artin and Schelter classified the Artin-Schelter regular algebras of global dimension 3, that is, graded associative algebras over \mathbb{C} with excellent homological properties. The algebras of type A with 3 generators in this classification are the Sklyanin algebras, with defining relations

$$\begin{aligned}axy + byx + cz^2 &= 0, \\ayz + bzy + cx^2 &= 0, \\azz + bxx + cy^2 &= 0,\end{aligned}$$

with some restrictions on $(a : b : c) \in \mathbb{P}^2$. In Artin et al. [2007], Artin, Tate and Van den Bergh showed that these Sklyanin algebras depend on an elliptic curve E and a point $\tau \in E$. It was noticed by Smith and Tate in Smith and Tate [1994] that the Heisenberg group H_3 of order 27 acts on such an algebra as gradation preserving automorphisms. In Odesskii and Feigin [1989], Odesskii and Feigin generalised the Sklyanin algebras to every dimension $n \geq 3$ and defined the n -dimensional Sklyanin algebras $A_n(\tau, E)$, where for each n H_n again works as algebra automorphisms on $A_n(\tau, E)$.

In those cases, $A_n(\tau, E)_1 \cong V = \mathbb{C}x_0 + \dots + \mathbb{C}x_n$ as an H_n -representation, with the action of H_n given by

$$e_1 \cdot x_i = x_{i-1}, e_2 \cdot x_i = \omega^i x_i,$$

ω being a primitive n th root of unity and the indices taken mod n . Since the Heisenberg group of order n^3 plays such an important role in the study of these Sklyanin algebras, other examples of graded Artin-Schelter regular algebras with n generators and on which H_n acts as gradation preserving automorphisms would be desired and in the best case, a complete classification. While this classification has been made for $n = 3$, a classification for $n \geq 4$ is not yet found.

Another interesting class of algebras is given by graded Clifford algebras, i.e. algebras depending on a quadratic form over a polynomial ring in an indeterminate number of variables, with entries of degree 2. Such algebras are always finite over their center and therefore they have a rich representation theory.

This paper looks at algebras that belong to these two worlds: it will discuss graded algebras $\mathfrak{C}(a, b)$, $(a, b) \in \mathbb{A}^2$ ($\mathfrak{C}(A : B : C)$, $(A : B : C) \in \mathbb{P}^2$), with generators in degree 1 and relations of the form

$$\begin{aligned} x_1x_4 + x_4x_1 &= ax_0^2, & x_2x_3 + x_3x_2 &= bx_0^2, \\ x_2x_0 + x_0x_2 &= ax_1^2, & x_3x_4 + x_4x_3 &= bx_1^2, \\ x_3x_1 + x_1x_3 &= ax_2^2, & x_4x_0 + x_0x_4 &= bx_2^2, \\ x_4x_2 + x_2x_4 &= ax_3^2, & x_0x_1 + x_1x_0 &= bx_3^2, \\ x_0x_3 + x_3x_0 &= ax_4^2, & x_1x_2 + x_2x_1 &= bx_4^2. \end{aligned}$$

We will call these H_5 -Clifford algebras (although these algebras aren't always Clifford algebras). Generically, a H_5 -Clifford algebra $\mathfrak{C}(a, b)$ will be a graded Clifford algebra generated in degree 1, with associated quadratic form

$$\begin{bmatrix} 2x_0^2 & bx_3^2 & ax_1^2 & ax_4^2 & bx_2^2 \\ bx_3^2 & 2x_1^2 & bx_4^2 & ax_2^2 & ax_0^2 \\ ax_1^2 & bx_4^2 & 2x_2^2 & bx_0^2 & ax_3^2 \\ ax_4^2 & ax_2^2 & bx_0^2 & 2x_3^2 & bx_1^2 \\ bx_2^2 & ax_0^2 & ax_3^2 & bx_1^2 & 2x_4^2 \end{bmatrix} \quad (1.1)$$

on the polynomial ring $\mathbb{C}[x_0^2, x_1^2, x_2^2, x_3^2, x_4^2]$. $\mathfrak{C}(a, b)_1$ will be isomorphic as an H_5 -representation to V_1 , where V_1 is the unique simple representation such that for $z = [e_1, e_2]$, $\varphi(z) = \omega I_5$ with $\omega = e^{\frac{2\pi i}{5}}$ and $\varphi : H_5 \rightarrow \mathbf{M}_5(\mathbb{C})$ the

morphism determined by V_1 . Crucial to describing these algebras will be the Koszul dual $\mathfrak{C}(a, b)^\dagger$, which will be a commutative algebra with 5 generators and 5 homogeneous relations and therefore it will define a projective variety in \mathbb{P}^4 .

Sometimes we will make the 10 relations we are interested in homogeneous, i.e. we will look at the algebras with equations

$$\begin{aligned} C(x_1x_4 + x_4x_1) &= Ax_0^2, & C(x_2x_3 + x_3x_2) &= Bx_0^2, \\ C(x_2x_0 + x_0x_2) &= Ax_1^2, & C(x_3x_4 + x_4x_3) &= Bx_1^2, \\ C(x_3x_1 + x_1x_3) &= Ax_2^2, & C(x_4x_0 + x_0x_4) &= Bx_2^2, \\ C(x_4x_2 + x_2x_4) &= Ax_3^2, & C(x_0x_1 + x_1x_0) &= Bx_3^2, \\ C(x_0x_3 + x_3x_0) &= Ax_4^2, & C(x_1x_2 + x_2x_1) &= Bx_4^2. \end{aligned}$$

and the following relations, which come naturally if $C = 1$, but must be included when $C = 0$.

$$\begin{aligned} B(x_1x_4 + x_4x_1) &= A(x_2x_3 + x_3x_2), \\ B(x_2x_0 + x_0x_2) &= A(x_3x_4 + x_4x_3), \\ B(x_3x_1 + x_1x_3) &= A(x_4x_0 + x_0x_4), \\ B(x_4x_2 + x_2x_4) &= A(x_0x_1 + x_1x_0), \\ B(x_0x_3 + x_3x_0) &= A(x_1x_2 + x_2x_1). \end{aligned}$$

Considering this, if we put $C = 0$, we still get an algebra with 10 quadratic relations, but we cannot hope to find Artin-Schelter regular algebras of global dimension 5 this way, since in these cases the algebras will not be of finite global dimension.

In the last section we will generalise certain phenomena in dimension 5 to H_p -Clifford algebras, p prime. The main result regarding quantum spaces will be

Theorem 1.1. *There are exactly $p + 1$ points (corresponding to the points in $\mathbb{P}_{\mathbb{F}_p}^1$) in $\mathbb{P}^{\frac{p-1}{2}}$ for which the corresponding H_p -Clifford algebra will be isomorphic to the quantum space $\mathbb{C}_{-1}[x_0, \dots, x_{p-1}]$.*

Considering the quantum spaces, we also have the following duality

Theorem 1.2. *There is a 1-to-1 correspondence between the $PSL_2(p)$ -orbit of the point $(1 : 0 : \dots : 0)$ and the PSL_2 -orbit of the line $a_0 = 0$. This correspondence is a morphism of $PSL_2(p)$ -sets and its action coincides with the canonical action of $PSL_2(p)$ on $\mathbb{P}_{\mathbb{F}_p}^1$.*

The main result regarding the regular H_p -Clifford algebras will be

Theorem 1.3. *The character series of a regular H_p -Clifford algebra is the same as the character series of the polynomial ring $\mathbb{C}[x_0, \dots, x_{p-1}]$, with the degree 1 part of the H_p -Clifford algebra isomorphic to the degree 1 part of $\mathbb{C}[x_0, \dots, x_{p-1}]$ as an H_p -representation.*

This means that, as an H_p -module, a H_p -Clifford algebra cannot be distinguished from the polynomial ring in p variables.

Notations

Throughout the paper, we will use the following notations and conventions

- $\mathbb{C}_{-1}[y_1, \dots, y_k]$ is the noncommutative algebra with defining relations $y_i y_j = -y_j y_i, 1 \leq i < j \leq k$.
- For a subset $S \subset \mathbb{C}[y_1, \dots, y_k]$, $\mathbf{V}(S)$ is the Zariski-closed subset defined by the elements of S . It will be clear from the context whether we look at subsets of \mathbb{A}^k or \mathbb{P}^{k-1} .
- μ_n is the set of n th roots of unity in \mathbb{C} .
- If G is a finite group and V is a G -representation, then the associated character of G is denoted by χ_V .

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2 Preliminaries

Those readers that are familiar with Koszul algebras, graded Clifford algebras, the representation theory of the Heisenberg group H_p with p prime and/or the connection between modular curves $X(p)$ and H_p may skip this section. The last part of the subsection Koszul algebras however will be unfamiliar to most readers.

2.1 Koszul algebras

Definition 2.1. *Given a quadratic algebra $A = T(V)/I$ with generators $V = \mathbb{C}x_0 + \dots + \mathbb{C}x_n$ and relations given by I_2 , we define the Koszul dual to be the quadratic algebra $T(V^*)/J$, with J_2 defined as the subspace of $V^* \otimes V^*$ such that $\forall w \in J_2, \forall v \in I_2 : w(v) = 0$.*

We say that A is Koszul iff $A^\dagger \cong \text{Ext}_A(\mathbb{C}, \mathbb{C})$. The standard properties of Koszul algebras we will need are that there is a relation between the Hilbert series of A and A^\dagger , given by

$$H_A(t)H_{A^\dagger}(-t) = 1$$

and that A is Koszul iff A^\dagger is Koszul.

An important fact concerning Koszul algebras is that the Koszul complex associated to these algebras is of the form

$$\dots \longrightarrow A \otimes (A^\dagger)_n^* \xrightarrow{(d_K)_n} A \otimes (A^\dagger)_{n-1}^* \longrightarrow \dots$$

and that

$$(A^\dagger)_n^* = V^{\otimes n-2} \otimes I_2 \cap \dots \cap I_2 \otimes V^{\otimes n-2}.$$

$(d_K)_n$ is given by taking the first component of $(A^\dagger)_n^*$ and absorbing it in A , for example $(d_K)_1(a \otimes x) = ax \in A$. It follows from this description that each $(d_K)_n$ is a G -morphism, whenever G acts on A as gradation preserving algebra automorphisms. This is useful for finding the character series when G is a finite (or more generally, reductive) group.

Definition 2.2. *Let G be a finite group. The character series for an element $g \in G$ and for a graded algebra A on which G acts as gradation preserving automorphisms is a formal sum*

$$Ch_A(g, t) = \sum_{n \in \mathbb{Z}} \chi_{A_n}(g) t^n.$$

For example, if $g = 1$, $Ch_A(1, t) = H_A(t)$, the Hilbert series of A . Since a character of a representation is constant on conjugacy classes, we can represent the decomposition of A in simple G -representations as a vector of length equal to the number of conjugacy classes and in the i th place the character series $Ch_A(g, t)$ with $g \in C_i$, the i th conjugacy class.

Suppose now that A is a Koszul algebra and that a finite group G acts on it as gradation preserving automorphisms. Because the Koszul complex is a free resolution of the trivial module \mathbb{C} , which is isomorphic as

G -representation to the trivial representation and because the Koszul complex consists of G -morphisms, we have a similar formula for finding the character series of the Koszul dual as we have for the Hilbert series. More precisely, we have

$$Ch_A(g, t)Ch_{(A^!)^*}(g, -t) = 1. \quad (2.1)$$

This allows us to compute $Ch_{A^!}(g, t)$ whenever we know $Ch_A(g, t)$. To know the character series of $A^!$, we have to take the complex conjugates of the coefficients of $Ch_{(A^!)^*}(g, t)$.

2.2 Graded Clifford algebras

This subsection will deal with the case we are interested in, but this is not the general definition. For more information, see Le Bruyn [1995] or Vancliff et al. [1998].

Given our algebra $\mathfrak{C}(a, b)$, we can associate to it 5 quadratic equations in the following way: we can write our equations as $x_i x_j + x_j x_i = (M_k)_{ij} x_k^2$ with $M_k \in \mathbf{M}_5(\mathbb{C})$. Taking $\{z_0, \dots, z_4\}$ as the basis of $(\mathfrak{C}(a, b)_1)^*$ such that $z_i(x_j) = \delta_{ij}$, we get 5 quadratic equations

$$q_k = [z_0, z_1, z_2, z_3, z_4] M_k \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}, k = 0, \dots, 4$$

This way, we get a quadric system. $\cap_{i=0}^4 \mathbf{V}(q_i)$ defines a Zariski closed set in $\mathbb{P}(\mathfrak{C}(a, b)_1)$ and a point in this closed set is called a base point. It is also clear that this set parametrizes the degree 1 elements of $\mathfrak{C}(a, b)$ whose square is 0, this follows from the definition of a Clifford algebra. The algebra $\mathbb{C}[z_0, z_1, z_2, z_3, z_4]/(q_0, q_1, q_2, q_3, q_4)$ is the Koszul dual $\mathfrak{C}(a, b)^!$ of $\mathfrak{C}(a, b)$.

In our case, the matrix M_0 for example is given by

$$M_0 = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & b & 0 \\ 0 & 0 & b & 0 & 0 \\ 0 & a & 0 & 0 & 0 \end{bmatrix}$$

and it follows that $q_0 = 2z_0^2 + 2az_1z_4 + 2bz_2z_3$. Since our relations are Heisenberg invariant, the other $q_k, k = 1, \dots, 4$ are easily found by cyclic permutation of the indices.

The next theorem will be crucial.

Theorem 2.3 (Cassidy and Vancliff [2010]). *A graded Clifford algebra \mathfrak{C} is quadratic, Auslander-regular of global dimension n , satisfies the Cohen-Macaulay property and has as Hilbert series $\frac{1}{(1-t)^n}$ if and only if the associated quadric system is base-point free. If this is the case, \mathfrak{C} is also a noetherian domain and Artin-Schelter regular.*

Remark 2.4. *While we will mainly work with graded Clifford algebras, we will also study quadratic algebras that are not Clifford algebras. These algebras will have as a quotient a graded Clifford algebra, the extra relations being implied by the fact that every square of an element of degree 1 is central in a graded Clifford algebra.*

2.3 The finite Heisenberg group(s)

Remark 2.5. *While a Heisenberg group of order n^3 can be defined for every $n \in \mathbb{N}$, the discussion here will only hold for $n = p$ prime. This is mainly to make things easier regarding the representation theory of these groups.*

Definition 2.6. *The Heisenberg group of order p^3 is the finite group given by the generators and relations*

$$H_p = \langle e_1, e_2, z | e_1^p = e_2^p = z^p, [e_1, e_2] = z, e_1 z = z e_1, e_2 z = z e_2 \rangle$$

and it is a central extension of the group $\mathbb{F}_p \times \mathbb{F}_p$

$$1 \longrightarrow \mathbb{F}_p \xrightarrow{1 \mapsto z} H_p \xrightarrow[e_2 \mapsto (0,1)]{e_1 \mapsto (1,0)} \mathbb{F}_p \times \mathbb{F}_p \longrightarrow 1 .$$

All the 1-dimensional simple representations of H_p are induced by the characters of $\mathbb{F}_p \times \mathbb{F}_p$. The other simple representations are p -dimensional and are determined by a primitive p th root of unity. They are defined in the following way: choose a primitive p th root of unity ω , then define the following action of H_p on the vector space $V = \mathbb{C}x_0 + \dots + \mathbb{C}x_{p-1}$

$$e_1 \cdot x_i = x_{i-1}, e_2 \cdot x_i = \omega^i x_i,$$

indices taken mod p . Taking another primitive root gives you another simple representation. This means that there are p^2 1-dimensional and $p-1$ p -dimensional irreducible representations, which are all the simple ones. There are $p^2 + p - 1$ conjugacy classes, 1 for each central element and the other $p^2 - 1$ classes contain a unique element of the form $e_1^a e_2^b$, $a, b \in \mathbb{F}_p$, $(a, b) \neq (0, 0)$.

The character of a simple p -dimensional representation V is given by

$$\begin{aligned}\chi_V(z^k) &= p\omega^k \\ \chi_V(e_1^a e_2^b) &= 0, (a, b) \neq (0, 0).\end{aligned}$$

Such a representation V also defines an antisymmetric bilinear form on the \mathbb{F}_p -vector space $\mathbb{F}_p \times \mathbb{F}_p$. Identifying e_1 and e_2 with their images in $\mathbb{F}_p \times \mathbb{F}_p$, we get this form by setting $\langle e_1, e_2 \rangle = \omega$ and extending it linearly to $\mathbb{F}_p \times \mathbb{F}_p$, thus

$$\langle ae_1 + be_2, ce_1 + de_2 \rangle = \omega^{ad-bc}.$$

If we define a group morphism $\langle z \rangle \xrightarrow{\phi} \mu_p$ by $\phi(z) = \omega$ (written multiplicatively in μ_p), then we have a commutative diagram

$$\begin{array}{ccc} H_p \times H_p & \longrightarrow & \mathbb{F}_p \times \mathbb{F}_p \\ \downarrow [\cdot] & & \downarrow \langle \cdot, \cdot \rangle \\ \langle z \rangle & \xrightarrow{\phi} & \mu_p \end{array}$$

Since every p -dimensional representation is determined by the image of z , every nontrivial antisymmetric bilinear form on $\mathbb{F}_p \times \mathbb{F}_p$ uniquely defines a simple representation of H_p . Conversely, every simple p -dimensional representation V of H_p defines a unique nontrivial antisymmetric bilinear form on $\mathbb{F}_p \times \mathbb{F}_p$ by extending linearly $\langle e_1, e_2 \rangle = \frac{\chi_V(z)}{p}$.

2.4 The modular curve $X(n)$

As is well known, the modular group $\Gamma = PSL_2(\mathbb{Z})$ acts on the complex upper half-plane

$$\mathbb{H} = \{x + iy | y > 0\}$$

by Möbius transformations. The fundamental domain of this action defines isomorphism classes of elliptic curves and its compactification, made by adding the Γ -orbit $\overline{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$, is the Riemann sphere S^2 . In general, one can take any other group G of finite index in Γ , find its fundamental domain in \mathbb{H} and check what information a point in this domain holds. The modular curve $X(n)$, $n \in \mathbb{N}$ is made this way by taking $G = \Gamma(n)$, with

$$\Gamma(n) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma \mid a, d \equiv 1 \pmod{n}, b, c \equiv 0 \pmod{n} \right\}.$$

A point on $X(n)$ holds 3 pieces of information:

- an elliptic curve (E, O) ,
- an embedding of $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ into E or equivalently, two generators e_1, e_2 of $E[n] = \{P \in E \mid [n]P = O\}$.
- a primitive n th root of unity ω such that $\langle e_1, e_2 \rangle = \omega$, where this bilinear antisymmetric form is found by the Weil-pairing.

$X(n)$ has an action of $PSL_2(n) = \Gamma/\Gamma(n)$ by definition. This action is defined by taking another set of generators of $E[n]$, f_1, f_2 , but their inner product must still remain ω . This defines an $SL_2(\mathbb{Z}/n\mathbb{Z})$ -action, but since $-I_2$ works trivially on $X(n)$, we have an $PSL_2(\mathbb{Z}/n\mathbb{Z})$ action.

Now let $n = p$ be prime. As we have seen in last subsection, a bilinear antisymmetric form on $\mathbb{F}_p \times \mathbb{F}_p$ defines a simple H_p -representation V . Let P_1, P_2 be a generating set of $E[p]$ and denote $P_{a,b} = [a]P_1 + [b]P_2$, then there exists a (unique up to multiplication with a scalar) function f on E with divisor

$$-(P_{0,0} + \dots + P_{0,p-1}) + P_{p-1,0} + \dots + P_{p-1,p-1}.$$

In Silverman [2009] it is proved that there exists a primitive p th root of unity such that $\omega = \frac{f}{\phi_{P_2}^*(f)}$, where ϕ_P^* stands for the pullback under the morphism

$$E \xrightarrow{\phi} E, \tau \mapsto \tau + P.$$

Calculating the divisor, one finds that the function

$$N(f) = f\phi_{P_1}^*(f) \dots (\phi_{P_1}^*)^{p-1}(f)$$

is constant and not 0, which means we can rescale f so that $N(f) = 1$. We will now define an action of H_p on the vector space

$$\mathcal{L}(P_{0,0} + \dots + P_{0,p-1}) = H^0(E, \mathcal{O}(P_{0,0} + \dots + P_{0,p-1})).$$

Let $x_0 = 1$ and define

$$\begin{aligned} e_1 \cdot g &= f\phi_{P_1}^*(g), \\ e_2 \cdot g &= \phi_{P_2}^*(g). \end{aligned}$$

If we set $x_i = e_1^{p-i} \cdot x_0$, we find that

$$\begin{aligned} e_1 \cdot x_i &= x_{i-1}, \\ e_2 \cdot x_i &= \omega^i x_i. \end{aligned}$$

This defines our action of H_p . These global sections define an embedding of E into \mathbb{P}^{p-1} and it is clear that the defining equations will be H_p -invariant.

3 The case $n = 3$

Before we start with the case $n = 5$, let us see what happens when $n = 3$. In this case, the classification of Artin-Schelter regular algebras has been done, see [Artin and Schelter, 1987] and [Artin et al., 2007].

When $n = 3$, the algebras we want to study have the following relations for $t \in \mathbb{C}$

$$xy + yx = tz^2, \quad yz + zy = tx^2, \quad zx + xz = ty^2.$$

Theorem 3.1. *For generic values of t , the H_3 -Clifford algebra $\mathfrak{C}(t)$ is a Sklyanin algebra associated to the elliptic curve*

$$E \leftrightarrow t(x^3 + y^3 + z^3) + (2 - t^3)xyz, O = (1, -1, 0)$$

and translation by the point $(1 : 1 : -t)$, which is a point of order 2.

There are however 7 values for $t \in \mathbb{C}$ where the corresponding H_3 -Clifford algebra is not a Sklyanin algebra: $t = 2, 2\omega, 2\omega^2, -1, -\omega, -\omega^2, 0$, with ω being a primitive third root of unity. The noncommutative algebra corresponding to $t = \infty$ is the algebra with relations

$$z^2 = x^2 = y^2 = 0$$

and is clearly not regular, so we have a total of 8 points in \mathbb{P}^1 where we don't have a Sklyanin algebra. When $t = 0, 2, 2\omega, 2\omega^2$, the corresponding algebra is still regular, but in the other 4 cases the Koszul dual $\mathfrak{C}(t)^\dagger$ does not define the empty set. For $t = -1, -\omega, -\omega^2, \infty$, $\mathfrak{C}(t)^\dagger$ defines a set of three points

$$\begin{aligned} t = -1 &\longleftrightarrow \{(1 : 1 : 1), (1 : \omega : \omega^2), (1 : \omega^2 : \omega)\}, \\ t = -\omega &\longleftrightarrow \{(1 : 1 : \omega^2), (1 : \omega : \omega), (1 : \omega^2 : 1)\}, \\ t = -\omega^2 &\longleftrightarrow \{(1 : 1 : \omega), (1 : \omega : 1), (1 : \omega^2 : \omega^2)\}, \\ t = \infty &\longleftrightarrow \{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)\}. \end{aligned}$$

Notice that we have an action of $PSL_2(3)$ on the moduli space \mathbb{P}^1 such that algebras in the same orbit are isomorphic, but the H_3 -representation is twisted (but still as an H_3 -module isomorphic to V_1): suppose that we have an algebra with relations

$$xy + yx = tz^2, \quad yz + zy = tx^2, \quad zx + xz = ty^2.$$

Then we can twist the Heisenberg action by an automorphism ϕ of H_3 and take the eigenvector with eigenvalue 1 of $\phi(e_2)$ and its orbit under $\phi(e_1)$ as

generators of this algebra (just the same as when $\phi = Id$, where x_0 is an eigenvector with eigenvalue 1 of e_2 and $x_i = e_1^{-i}x_0$). We want the twist to preserve the character of the representation, which means that we need to preserve the antisymmetric inner product on $\mathbb{F}_3 \times \mathbb{F}_3$ defined by $\langle e_1, e_2 \rangle = \omega$ (written multiplicatively), where e_1 and e_2 are identified with their images in $\mathbb{F}_3 \times \mathbb{F}_3$ like in the previous section.

The condition $\langle \phi(e_1), \phi(e_2) \rangle = \omega$ means that we need to look at elements of $SL_2(3)$. The element $-I_2$ however will work trivially on \mathbb{P}^1 , so it will be a $PSL_2(3)$ -action. Using the following generators

$$U = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, V = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix},$$

with $U^2 = V^3 = I$, the action of $PSL_2(3) \cong A_4$ is given by the following Möbius transformations:

$$U \leftrightarrow U'(t) = \frac{-t+2}{t+1}, V \leftrightarrow V'(t) = \frac{-\omega^2 t + 2}{\omega^2 t + 1}.$$

This action is a *right* action, for this action comes naturally by twisting the group morphism $\varphi : H_3 \rightarrow Aut(\mathfrak{C}(t))$ by an automorphism of H_3 . So suppose that we have an automorphism $\psi : H_3 \rightarrow H_3$, then we get a new group morphism given by $\varphi \circ \psi : H_3 \rightarrow Aut(\mathfrak{C}(t))$.

Using this action, the sets $\{0, 2, 2\omega, 2\omega^2\}$ and $\{\infty, -1, -\omega, -\omega^2\}$ correspond to the action of $PSL_2(3)$ on $\mathbb{P}_{\mathbb{F}_3}^1$.

Example 3.2. *Suppose we want to calculate the action of*

$$M = VU = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

on our moduli space \mathbb{P}^1 . Taking our algebra $\mathfrak{C}(t)$ (and we allow $t = \infty$), this means we need to find the eigenvector of $M \cdot e_2$ with eigenvalue 1. In this particular case, this eigenvector is still x_0 , but now x_2 is defined as $(e_1 e_2)x_0$ instead of $e_1 x_0$. So if we take y_0, y_1, y_2 as the generators of our isomorphic algebra, we have

$$\begin{aligned} y_0 &= x_0, \\ y_1 &= \omega^2 x_1, \\ y_2 &= x_2. \end{aligned}$$

A quick calculation shows that $y_0 y_1 + y_1 y_0 = \omega^2 t y_2^2$ and so the corresponding Möbius transformation is given by $t \mapsto \omega^2 t$, with fixed points 0 and ∞ . This

shows it is a right action: the composition $U' \circ V'$ is equal to the action of M , but $M = VU$.

In this case, the action of $PSL_2(3)$ is not a projectification of a 2-dimensional representation, because all $PSL_2(3)$ -representations of dimension 2 would be direct sums of 1-dimensional representations. This is impossible because then every element of order 2 would act trivially and this is clearly not the case.

One other thing of importance is that we have a duality between the quantum spaces (the algebras isomorphic to the algebra $\mathfrak{C}(0)$) and the 4 nonregular algebras, given by the point varieties of the regular ones and the Koszul dual of the nonregular ones. Before we give this duality, we recall the definition of a point module.

Definition 3.3. *Let B be a positively graded, connected \mathbb{C} -algebra generated in degree 1. A point module M is a cyclic, graded left B -module with Hilbert series $\frac{1}{1-t}$ with $M = BM_0$.*

Example 3.4. *For the quantum space $\mathbb{C}_{-1}[x_0, \dots, x_{p-1}]$, the point modules are described by the full graph on p points, the p vertices given by the H_p -orbit of the point $(1 : 0 : \dots : 0)$.*

The duality we want to show is given in the following way: let $\mathfrak{C}(t_1)$ be an algebra isomorphic to the quantum plane. Then its point variety (the point modules of this algebra) is given by the full graph on 3 points, the 3 points corresponding to a unique H_3 -orbit. Then there exists a unique nonregular algebra $\mathfrak{C}(t_2)$ in our moduli space such that its Koszul dual $\mathfrak{C}(t_2)^\dagger$ has as its point variety this unique H_3 -orbit. For example, the point variety of the quantum plane $\mathfrak{C}(0)$

$$xy + yx = xz + zx = yz + zy = 0$$

is given by the three lines $x = 0, y = 0, z = 0$, and they intersect 2 by 2 in 3 points. These points are the solutions of the equations $xy = xz = yz = 0$. These equations are the relations of the Koszul dual of the algebra with relations

$$x^2 = y^2 = z^2 = 0,$$

which is the algebra $\mathfrak{C}(\infty)$.

Theorem 3.5. *There is a 1-to-1 correspondence between the H_3 -Clifford algebras isomorphic to $\mathbb{C}_{-1}[x_0, x_1, x_2]$ and the nonregular H_3 -Clifford algebras, with the correspondence being given in the following way: the regular*

H_3 -Clifford algebra $\mathfrak{C}(t)$ corresponds to the nonregular $\mathfrak{C}(t')$ iff t and t' are fixed by the same subgroup of order 3 in $PSL_2(3)$. This correspondence gives a natural bijection between $\mathbb{P}_{\mathbb{F}_3}^1$ and the points in \mathbb{P}^1 with corresponding H_3 -Clifford algebra isomorphic to the quantum plane.

Remark 3.6. Although 4 is not prime, one can ask the natural question what happens if $n = 4$. This case is however not very interesting: there aren't any quadratic algebras \mathfrak{C} with defining relations $x_i x_j + x_j x_i = a_{ij} x_k^2$ and with degree 1 part isomorphic to V_1 , V_1 being the H_4 representation with z working as iI_4 (apart from the quantum space with $a = 0$). There aren't any squares of elements in V_1 on which e_2 works as multiplying by $\pm i$, so this forces that $x_i x_j + x_j x_i = 0$ whenever $j - i \equiv 1 \pmod{2}$. When $j - i \equiv 0 \pmod{2}$, we need to define $x_0 x_2 + x_2 x_0$ and $x_1 x_3 + x_3 x_1$, so we need to find elements v and w in degree 1 such that the action on v^2 and w^2 by e_2 is respectively given by multiplication with -1 and 1 , with the extra condition that e_1 permutes v and w . Such elements are impossible to find (except when $u = v = 0$), so we are done.

By the last remark we have proved

Theorem 3.7. The only H_4 -Clifford algebra is $\mathbb{C}_{-1}[x_0, x_1, x_2, x_3]$, which is regular.

4 The case $n = 5$

From now on, $\omega = e^{\frac{2\pi i}{5}}$. Similar to the case of $n = 3$, we have a right action of $PSL_2(5) \cong A_5$ on \mathbb{P}^2 that gives isomorphic algebras in the moduli space. This action is found using the exact same procedure as in the previous section. Again using U, V as generators of $PSL_2(5)$, we get the following matrices

$$U \leftrightarrow \frac{1}{\sqrt{5}} \begin{bmatrix} \omega^2 + \omega^3 & \omega + \omega^4 & 2 \\ \omega + \omega^4 & \omega^2 + \omega^3 & 2 \\ 1 & 1 & 1 \end{bmatrix}, V \leftrightarrow \frac{1}{\sqrt{5}} \begin{bmatrix} \omega + \omega^2 & \omega^2 + 1 & 2 \\ \omega^3 + 1 & \omega^3 + \omega^4 & 2 \\ \omega^4 & \omega & 1 \end{bmatrix}$$

and now this projective representation comes from a simple representation of A_5 , the icosahedron representation. The factor $\frac{1}{\sqrt{5}}$ is necessary to get an A_5 representation, but it doesn't matter for our algebras. To discuss these algebras, we will sometimes use the algebras which correspond to points at infinity, because their equations are easier to work with.

4.1 The nonregular algebras

The nicest algebras are those that are noetherian domains, so to find them we will use theorem 2.3. Using this theorem and the fact that the equations of the quadric system correspond to the relations of the Koszul dual, we need to analyse the following equations

$$x_0^2 + ax_1x_4 + bx_2x_3 = 0, \quad (4.1a)$$

$$x_1^2 + ax_2x_0 + bx_3x_4 = 0, \quad (4.1b)$$

$$x_2^2 + ax_3x_1 + bx_4x_0 = 0, \quad (4.1c)$$

$$x_3^2 + ax_4x_2 + bx_0x_1 = 0, \quad (4.1d)$$

$$x_4^2 + ax_0x_3 + bx_1x_2 = 0 \quad (4.1e)$$

and determine when the only solution is given by $(0, 0, 0, 0, 0)$.

We can also look at the H_5 -Clifford algebras corresponding to points on the line at infinity, so that we get a projective variety for every point in our moduli space \mathbb{P}^2 . We do this by changing (a, b) to $(A : B : C)$ and putting a C before every x_i^2 . However, since every point on the line at infinity of \mathbb{P}^2 is in the $PSL_2(5)$ -orbit of an affine point, this is not always interesting. The points at infinity however will give equations that are easier to handle in some cases.

Theorem 4.1. *Generically, the H_5 -Clifford algebra $\mathfrak{C}(A : B : C)$ is a regular graded Clifford algebra.*

Proof. Calculating the Gröbner basis of the relations of $\mathfrak{C}(a, b)$ ¹ using Mathematica, one finds that the following monomial belongs to the ideal I defined by the equations from 4.1

$$(1 + a^5 - 4ab + a^6b + 5a^3b^3 + b^5 + ab^6)x_4^6. \quad (4.2)$$

This means that, if $1 + a^5 - 4ab + a^6b + 5a^3b^3 + b^5 + ab^6 \neq 0$, $x_4^6 \in I$. But I is closed under the action of the Heisenberg group, so this implies that every x_i^6 belongs to I . This implies that I defines the empty set and so $\mathfrak{C}(a, b)$ would have all the good properties we desire. \square

So generically, we get a regular graded Clifford algebra. The only possible ‘bad’ algebras are given by the curve $1 + a^5 - 4ab + a^6b + 5a^3b^3 + b^5 + ab^6 = 0$. Adding the line $C = 0$, decomposing the equation $1 + a^5 - 4ab + a^6b + 5a^3b^3 +$

$b^5 + ab^6 = 0$ and looking at the projective closure of this variety, we get 6 lines and a conic section

$$C = 0, \quad (4.3a)$$

$$C + A + B = 0, \quad (4.3b)$$

$$C + \omega A + \omega^4 B = 0, \quad (4.3c)$$

$$C + \omega^4 A + \omega B = 0, \quad (4.3d)$$

$$C + \omega^2 A + \omega^3 B = 0 \quad (4.3e)$$

$$C + \omega^3 A + \omega^2 B = 0, \quad (4.3f)$$

$$AB + C^2 = 0, \quad (4.3g)$$

where the regularity condition possibly fails. Notice that the 6 lines we found form an orbit under $PSL_2(5)$ (as we expected). This means that, if we want to analyse how ‘far’ these algebras are from being Artin-Schelter regular, we can restrict ourselves to studying the points on the line $C = 0$ and on the conic section $AB + C^2$. For the line $C = 0$, this boils down to describing the relations of the form

$$Ax_1x_4 + Bx_2x_3 = 0, \quad (4.4a)$$

$$Ax_2x_0 + Bx_3x_4 = 0, \quad (4.4b)$$

$$Ax_3x_1 + Bx_4x_0 = 0, \quad (4.4c)$$

$$Ax_4x_2 + Bx_0x_1 = 0, \quad (4.4d)$$

$$Ax_0x_3 + Bx_1x_2 = 0, \quad (4.4e)$$

excluding the case $(A, B) = (0, 0)$. For a generic point on the line $C = 0$, one can use Macaulay2 and find that the Hilbert series is given by

$$\frac{1 + 4t + 5t^2 - 5t^4}{1 - t}.$$

So it follows that generically, the H_5 -orbit of $(1 : 0 : 0 : 0 : 0)$ is the solution set of the equations from 4.4. However, when you calculate the Gröbner basis (which determines the Hilbert series) using Mathematica, there are 7 points on this line where this (possibly) fails:

$$(A : B : C) = (1 : -\omega^k : 0), k = 0 \dots 4,$$

$$(A : B : C) = (1 : 0 : 0),$$

$$(A : B : C) = (0 : 1 : 0).$$

The Hilbert series indeed fails here. Calculating Hilbert series, we find

Theorem 4.2. *For a point on the line $C = 0$, the algebra $\mathfrak{C}(A : B : C)^!$ determines the H_5 -orbit of $(1 : 0 : 0 : 0 : 0)$ as Zariski-closed subset of \mathbb{P}^4 , except in the following 7 cases:*

- *For $k = 0, \dots, 4$, the point $(1 : -\omega^k : 0)$ is the intersection of the line $C = 0$ and the line $C + \omega^{3k}A + \omega^{-3k}B = 0$. At the point $(1 : -\omega^k : 0)$, the projective variety determined by $\mathfrak{C}(1 : -\omega^k : 0)^!$ is given by 10 points. These 10 points form the union of 2 H_5 -orbits, each orbit consisting of 5 elements. Representatives of these 2 orbits are given by $(1 : 0 : 0 : 0 : 0)$ and $(1 : 1 : \omega^{-2k} : \omega^{-k} : \omega^{-2k})$. The Hilbert series of the commutative algebra $\mathfrak{C}(A : B : C)^!$ is given by*

$$\frac{1 + 4t + 5t^2}{1 - t}.$$

- *For $(A : B : C) = (1 : 0 : 0)$, we get 5 lines, $\cup_{i=0}^4 \mathbf{V}(x_i, x_{i+1}, x_{i+2})$. The Hilbert series $\mathfrak{C}(1 : 0 : 0)^!$ is given by*

$$\frac{1 + 3t + t^2}{(1 - t)^2}.$$

The corresponding configuration with its vertices the 5 points in the orbit of $(1 : 0 : 0 : 0 : 0)$ is given by figure 1.

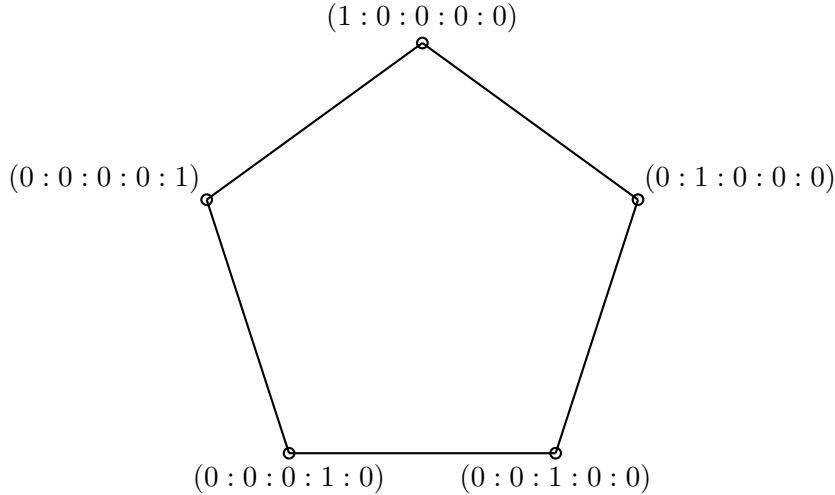


Figure 1: First configuration

- For $(A : B : C) = (0 : 1 : 0)$, there are again 5 lines, now given by $\cup_{i=0}^4 \mathbf{V}(x_i, x_{i+1}, x_{i+3})$. The Hilbert series of $\mathfrak{C}(0 : 1 : 0)^1$ is given by

$$\frac{1 + 3t + t^2}{(1 - t)^2}.$$

The corresponding configuration is given by figure 2.

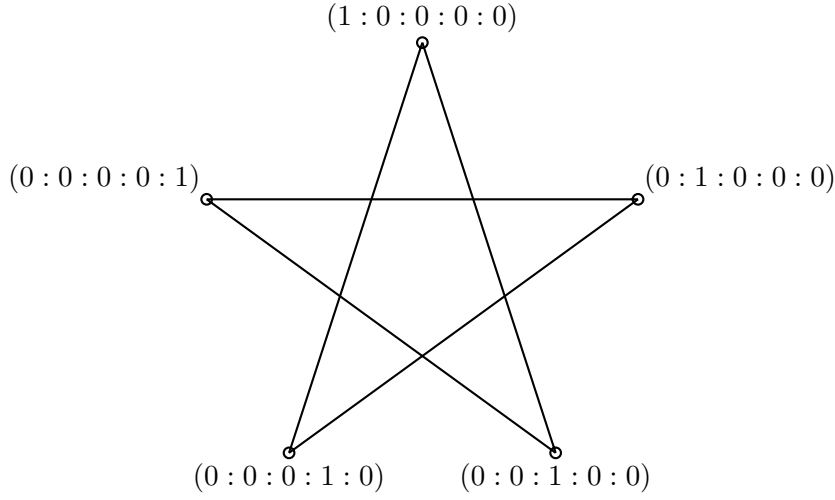


Figure 2: Second configuration

For a point on the conic section $AB + C^2$, the relations from equation 4.1 determine a smooth genus 1 curve, except when $A = 0$, $B = 0$ or

$$\begin{aligned} (A : B : C) &= (\omega^k(\omega^2 + \omega^3), \omega^{-k}(\omega + \omega^4), 1), k = 0, \dots, 4, \\ (A : B : C) &= (\omega^k(\omega + \omega^4), \omega^{-k}(\omega^2 + \omega^3), 1), k = 0, \dots, 4. \end{aligned}$$

These 12 points are of course the intersection points with the 6 lines from equation 4.3. For every affine point $(a, \frac{-1}{a})$ excluding the 12 special points, the point $(0 : 1 : a : -a : -1)$ and its H_5 -orbit lies on the curve defined by 4.1. Taking this point to be the point O , the corresponding curve is an elliptic curve E , and the H_5 -orbit of O gives an embedding of $\mathbb{F}_5 \times \mathbb{F}_5$ in E . It follows from calculations using Mathematica and Macaulay2 that every point on $\mathbf{V}(AB + C^2)$ determines an elliptic curve, except for 12 points, which are

the intersection points of the 6 lines from equation 4.3. A $PSL_2(5)$ -orbit on this conic sections determines an isomorphism class of elliptic curves, but changes the chosen generators of $E[5]$. These new generators however have the same Weil pairing as the original generators (as they determine the representation in degree 1, which is unchanged) and therefore the conic section $\mathbf{V}(AB + C^2) - \{12 \text{ points}\}$ with the $PSL_2(5)$ -action is a model for the modular curve $X(5)$. The extra 12 points give the compactification of $X(5)$. Summarizing, we have the following theorem:

Theorem 4.3. *For the points on the curve $\mathbf{V}(AB + C^2) - \{12 \text{ points}\}$, the corresponding algebra $\mathfrak{C}(A : B : C)^!$ is the homogeneous coordinate ring of an elliptic curve E embedded in \mathbb{P}^4 with $O = (0 : 1 : a : -a : -1)$. Furthermore, the point $(A : B : C)$ determines an embedding of $\mathbb{F}_5 \times \mathbb{F}_5$ in E and a fixed \mathbb{F}_5 -basis $(e_1 \cdot O, e_2 \cdot O)$. Conversely, every point in $X(5)$ determines a unique point on $\mathbf{V}(AB + C^2)$ and thus $\mathbf{V}(AB + C^2)$ is a model for $X(5)$, with the 12 extra points the points needed to make the compactification $X(5)$.*

Now, choose a line from equations 4.3. This line will intersect $\mathbf{V}(AB + C^2)$ in exactly 2 points, P_1 and P_2 . The commutative algebra corresponding to P_1 will describe the union of 5 lines that intersect 2 by 2. The intersection points will be a H_5 -orbit of 5 elements and the configuration will be like figure 1. The point P_2 will also determine the union of 5 lines that intersect 2 by 2, with the intersection points the same as for P_1 , but now the configuration will be like figure 2.

4.2 Being Koszul

This section gives a proof of the following theorem.

Theorem 4.4. *Every H_5 -Clifford algebra $\mathfrak{C}(A : B : C)$ is Koszul, except for the points on the 6 lines given by the $PSL_2(5)$ -orbit of $C = 0$ that do not lie on the conic section $\mathbf{V}(AB + C^2)$.*

Proof. When $\mathfrak{C}(A : B : C)$ is regular (the generic case), we can use theorem 2.3 and apply theorem 2.2 of Shelton and Tingey [2001] to conclude that $\mathfrak{C}(A : B : C)$ is regular. When the Koszul dual is an elliptic curve, it follows from the Koszulity of the homogeneous coordinate ring of an elliptic curve that the algebras of the form (choosing representatives such that the sum of the indices is 0)

$$x_1x_4 + x_4x_1 = ax_0^2, \quad x_1x_4 + x_4x_1 = \frac{-1}{a}x_0^2$$

are indeed Koszul whenever the point $(a, \frac{-1}{a})$ determines an elliptic curve. For the 12 points on $\mathbf{V}(AB + C^2)$ that determine 5 lines, Koszulity follows from the fact that in this case $\mathfrak{C}(A : B : C)$ is isomorphic to $\mathfrak{C}(1 : 0 : 0)$. The relations of $\mathfrak{C}(1 : 0 : 0)^1$ are given by 5 monomials of degree 2 (and the commutator relations of course). Then theorem 3.15 of [Conca et al., 2013] gives us that $\mathfrak{C}(1 : 0 : 0)^1$ is indeed Koszul, but then $\mathfrak{C}(1 : 0 : 0)$ is also Koszul.

These are all the Koszul algebras. Assume that $(A : B : C)$ lies on 1 of the 6 lines from 4.1 (excluding the 12 points lying on the conic section and the 15 points at the intersections) and that $\mathfrak{C}(A : B : C)$ is Koszul. Then $\mathfrak{C}(A : B : C)$ has as Hilbert series

$$\frac{1+t}{1-4t+5t^2-5t^4} = 1 + 5t + 15t^2 + 35t^3 + 70t^4 + 130t^5 + \dots$$

This would mean that the Hilbert series of this algebra is equal to $\frac{1}{(1-t)^5}$ up to degree 4. But then it follows from the fact that this algebra has as a quotient a graded Clifford algebra that it should have Hilbert series $\frac{1}{(1-t)^5}$, which is not the case.

The same argument also applies when the Koszul dual of $\mathfrak{C}(A : B : C)$ determines 10 points, because now its Hilbert series should be

$$\frac{1+t}{1-4t+5t^2} = 1 + 5t + 15t^2 + 35t^3 + 65t^4 + 85t^5 + \dots$$

and this is clearly not $\frac{1}{(1-t)^5}$. □

4.3 The quantum planes and their relationship with the 6 lines

$PSL_2(5)$ has 6 5-Sylow groups, each one stabilizing one of the 6 lines of equation 4.3. However, such a group doesn't fix the entire line, but rather 2 points (the intersection with $\mathbf{V}(AB + C^2)$). Since these groups are cyclic and they are all conjugated, it suffices to work with one 5-Sylow group, e.g. the group generated by VU . The corresponding matrix is given by

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} \omega^4 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so the third point fixed by this matrix is $(0 : 0 : 1)$, with corresponding algebra the quantum space $\mathfrak{C}(0 : 0 : 1)$ with relations $x_k x_l + x_l x_k = 0, k \neq l$.

The other points in the $PSL_2(5)$ -orbit of $(0 : 0 : 1)$ are given by $(2\omega^k : 2\omega^{-k} : 1), k = 0, \dots, 4$. This gives us 6 points with corresponding Clifford algebra isomorphic to the quantum space $\mathfrak{C}(0 : 0 : 1)$, with the $PSL_2(5)$ -action the same as its action on $\mathbb{P}_{\mathbb{F}_5}^2$.

Again we have a certain duality regarding these quantum algebras and the 6 lines from equation 4.3. I will explain it using the example of $(0 : 0 : 1)$ and the line $C = 0$. The point modules of the quantum plane in this case are parametrized by the full graph on 5 points, with its vertices given by the H_5 -orbit of $(1 : 0 : 0 : 0 : 0)$. The configuration is given by figure 3.

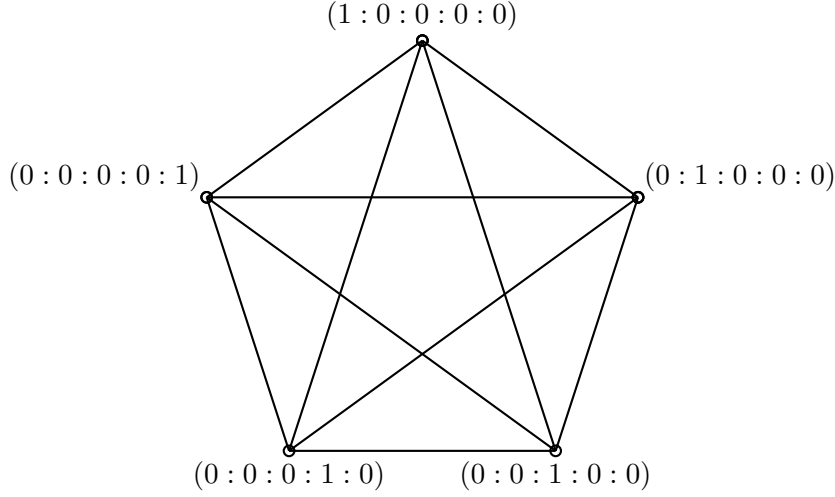


Figure 3: The full graph on 5 points

The graph of figure 3 is the union of figure 1 and 2, which were projective varieties determined by the $\mathfrak{C}(1 : 0 : 0)^!$ and $\mathfrak{C}(0 : 1 : 0)^!$. More generally, every point on the line $C = 0$ has the H_5 -orbit of $(1 : 0 : 0 : 0 : 0)$ in the zeroset of $\mathfrak{C}(A : B : C)^!$, which are the vertices of the graph of figure 3. Summarizing, we have

Theorem 4.5. *There is a 1-to-1 correspondence between the $PSL_2(5)$ -orbit of $(0 : 0 : 1)$ and the $PSL_2(5)$ -orbit of the line $C = 0$. Moreover, for every line in the $PSL_2(5)$ -orbit of the line $C = 0$, there are 2 points, the intersections with $\mathbf{V}(AB + C^2)$, whose Koszul dual determines a graph on 5 points. The union of these 2 graphs gives the full graph on 5 points, which*

is the point variety of the corresponding algebra isomorphic to the quantum space.

The 2 points of $\mathbf{V}(AB + C^2)$ that correspond to the same quantum space are determined by the fact that they are fixed by the same cyclic subgroup of order 5 in $PSL_2(5)$.

5 Generalities for prime dimension

In this section, $\omega = e^{\frac{2\pi i}{p}}$ with $p \geq 5$ prime. Many things described in the previous section for the case $p = 5$ can be arbitrarily extended to every dimension p , p prime. The relations we are now interested in are given by the H_p -action on (indices are taken mod p)

$$a_0(x_0x_i + x_ix_0) = a_ix_{\frac{i}{2}}^2, 0 < i \leq \frac{p-1}{2},$$

so our moduli space is given by $\mathbb{P}^{\frac{p-1}{2}}$ and again we have a $PSL_2(p)$ action on this space.

5.1 The quantum spaces

We will prove here the following theorem.

Theorem 5.1. *There are exactly $p + 1$ points in $\mathbb{P}^{\frac{p-1}{2}}$ for which the corresponding algebra $\mathfrak{C}(a_0 : \dots : a_{\frac{p-1}{2}})$ is isomorphic to the algebra with relations $x_ix_j + x_jx_i = 0, i \neq j$ and they form an orbit under the $PSL_2(p)$ -action.*

Before we prove this, some considerations must be made. First of all, the elements of \mathbb{P}^{p-1} with a nontrivial stabilizer in H_p must be found. This is equivalent to finding the eigenvectors of all the elements of H_p . Since a central element has every vector as eigenvector (and works therefore as the identity on \mathbb{P}^{p-1}), it is sufficient to consider elements of the form $e_1^k e_2^l$. Since $e_1 e_2 = z e_2 e_1$ with z central, we can take powers of $(e_1^k e_2^l)^m$ until we get something of the form $z^r e_1^{-1} e_2^t$, unless $k = 0$. Disregarding the part z^r , it suffices to determine the eigenvectors for the elements $e_1^{-1} e_2^t, t = 0, \dots, p-1$ and the element e_2 . Of course, points belonging to the same orbit have the same stabilizer (since the H_p -action on \mathbb{P}^{p-1} is actually an action of $\mathbb{F}_p \times \mathbb{F}_p$ and this group is commutative), so it is sufficient to give one representative of such an orbit. This means we have a total of $p + 1$ orbits in \mathbb{P}^{p-1} with the property that such an orbit consists of p elements instead of p^2 elements.

Lemma 5.2. *The fixed points of $e_1^{-1}e_2^k, k = 0, \dots, p-1$ are given by the H_p -orbit of the following element:*

$$(1 : \omega^k : \omega^{3k} : \dots : \omega^{k\frac{p(p-1)}{2}}),$$

which has as its i th coordinate $\omega^{k\frac{i(i+1)}{2}}$, if we start counting from 0. The fixed points of e_2 are given by the H_p -orbit of $(1 : 0 : \dots : 0)$. These are all the points with a nontrivial stabilizer.

Proof. We again identify the image of e_1 and e_2 in $\mathbb{F}_p \times \mathbb{F}_p$ with the generators of H_p . Since the order of $\mathbb{F}_p \times \mathbb{F}_p$ is p^2 , every point in \mathbb{P}^{p-1} has as its stabilizer the entire group, nothing, or a cyclic group of order p . The entire group is impossible, since the action of H_p on \mathbb{C}^p was irreducible. Every cyclic group of order p in $\mathbb{F}_p \times \mathbb{F}_p$ has a generator of the form e_2 or $e_1^{-1}e_2^k, k = 0, \dots, p-1$. Since the eigenvalues of the matrices of the representation are all distinct, there are exactly p points that are fixed by any such element. So if we check that the element claimed by the theorem indeed is fixed by $e_1^{-1}e_2^k$, we are done (the claim for the group generated by e_2 is trivial). By a calculation, we get

$$\begin{aligned} & e_1^{-1}e_2^k(1 : \dots : \omega^{k\frac{i(i-1)}{2}} : \omega^{k\frac{i(i+1)}{2}} : \dots : \omega^{k\frac{p(p-1)}{2}}) \\ &= e_1^{-1}(1 : \dots : \omega^{k\frac{i(i-1)}{2} + (i-1)k} : \omega^{k\frac{i(i+1)}{2} + ik} : \dots : \omega^{k\frac{p(p-1)}{2} + k(p-1)}) \\ &= (\omega^{k\frac{p(p-1)}{2} + k(p-1)} : \dots : \underbrace{\omega^{k\frac{i(i-1)}{2} + (i-1)k}}_i : \underbrace{\omega^{k\frac{i(i+1)}{2} + ik}}_{i+1} : \dots : \omega^{k\frac{(p-2)(p-1)}{2} + k(p-2)}) \end{aligned}$$

Now, we have that $\omega^{k\frac{p(p-1)}{2} + k(p-1)} = \omega^{-k}$, so we may multiply each coordinate with ω^k and this gives us

$$= (1 : \dots : \underbrace{\omega^{k\frac{i(i+1)}{2}}}_i : \underbrace{\omega^{k\frac{(i+2)(i+1)}{2}}}_{i+1} : \dots : \omega^{k\frac{p(p-1)}{2}}).$$

□

Calculating what the action of U on $(1 : 0 : \dots : 0) \in \mathbb{P}^{\frac{p-1}{2}}$ in the moduli space exactly is, we see that U switches the point $(1 : 0 : \dots : 0)$ with $(1 : 2 : \dots : 2)$. Next, the action of VU fixes the point $(1 : 0 : \dots : 0)$, but doesn't fix the point $(1 : 2 : \dots : 2)$. Since the order of VU is prime, this means that the VU -orbit of $(1 : 2 : \dots : 2)$ consists of p elements. Combining these 2 observations, we have

Lemma 5.3. *The $PSL_2(p)$ -orbit of $(1 : 0 : \dots : 0)$ consists of at least $p + 1$ elements.*

Proof. [Theorem 5.1] The quantum space $\mathfrak{C}(1 : 0 : \dots : 0 : 0)$ has the following property: there are exactly p points in \mathbb{P}^{p-1} for which the associated quadratic form has rank 1 and these p points form an orbit under the Heisenberg action. So every algebra determined by a point in our moduli space $\mathbb{P}^{\frac{p-1}{2}}$ and isomorphic to the quantum space must have the same property. In 5.2 we found that there are $p + 1$ different orbits consisting of p points. Let $\rho = \omega^2$. We may set $a_0 = 1$, because otherwise the algebra $\mathfrak{C}(a_0 : \dots : a_{\frac{p-1}{2}})$ is not a domain and therefore it can not be isomorphic to $\mathfrak{C}(1 : 0 : \dots : 0 : 0)$. So if the algebra $\mathfrak{C}(a_1, \dots, a_{\frac{p-1}{2}})$ is isomorphic to $\mathfrak{C}(1 : 0 : \dots : 0 : 0) = \mathfrak{C}(0, \dots, 0)$, we must have that the matrix

$$\begin{bmatrix} 2x_0^2 & a_1x_{\frac{p+1}{2}}^2 & a_2x_1^2 & \cdots & a_{\frac{p-1}{2}}x_{\frac{p+\frac{p-1}{2}}{2}}^2 & \cdots & a_1x_{\frac{p+(p-1)}{2}}^2 \\ a_1x_{\frac{p+1}{2}}^2 & 2x_1^2 & a_1x_{\frac{p+3}{2}}^2 & \cdots & \cdots & \cdots & a_2x_0^2 \\ \vdots & \vdots & \ddots & & & & \end{bmatrix}$$

has rank 1 when $(x_0^2, \dots, x_{p-1}^2)$ is equal to one of the points found in 5.2 (with ω changed by ρ since z acts on the representation $\mathbb{C}x_0^2 + \dots + \mathbb{C}x_{p-1}^2$ as ρI_p). These conditions completely determine $a_1, \dots, a_{\frac{p-1}{2}}$ and therefore we can maximally have $p + 1$ algebras isomorphic to $\mathfrak{C}(1 : 0 : \dots : 0 : 0)$. But lemma 5.3 gives us that there are minimally $p + 1$ points, so we have exactly $p + 1$ points. Therefore, the $PSL_2(p)$ -orbit of $(1 : 0 : \dots : 0) \in \mathbb{P}^{\frac{p-1}{2}}$ gives all the algebras isomorphic to the quantum space and this orbit consists of $p + 1$ points. \square

5.2 Generalizing the duality

We have a refinement of the duality between the quantum space and $\frac{p-1}{2}$ nonregular algebras, the algebras $\mathfrak{C}(0 : 0 : \dots : \underbrace{1}_i : \dots : 0), i \neq 0$. This is established in the following way: the algebra $\mathfrak{C}(0 : \dots : \underbrace{1}_i : \dots : 0 : 0)$ can be recovered from the quantum plane by deleting the H_p -orbit of $a_0(x_0x_i + x_ix_0) = a_ix_i^2$ and instead adding the relations $x_0^2 = x_1^2 = \dots = x_{p-1}^2 = 0$. The Koszul dual of this algebra is a commutative algebra with relations $x_0x_i = x_1x_{i+1} = \dots = x_{p-1}x_{p-1+i} = 0$. These relations determine p linear spaces of dimension $\frac{p-1}{2} - 1$ and for every $0 < i \leq \frac{p-1}{2}$ the corresponding

union of p -dimensional subspaces is different from the other ones. It also follows that the points $(0 : 0 : \dots : \underbrace{1}_i : \dots : 0) \in \mathbb{P}^{\frac{p-1}{2}}$ are fixed by the element

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \in PSL_2(p).$$

Another duality is given between the $p+1$ hyperplanes in $\mathbb{P}^{\frac{p-1}{2}}$ determined by the $PSL_2(p)$ -orbit of the hyperplane $a_0 = 0$ and the $p+1$ algebras isomorphic to the quantum space. Generically, a point on such a hyperplane determines a unique H_p -orbit of p points. These p points are the vertices of the full graph on these p points, which will parametrize the point variety of the corresponding algebra isomorphic to the quantum plane. So we have the following theorem

Theorem 5.4. *There is a 1-to-1 correspondence between the algebras isomorphic to the quantum space and the $PSL_2(p)$ -orbit of the hyperplane $a_0 = 0$. This correspondence is given by the following rule: take the unique p -Sylow group of $PSL_2(p)$ that fixes the point in the $PSL_2(p)$ -orbit of $(1 : 0 : \dots : 0)$. Then there are $\frac{p-1}{2}$ other points that are also fixed by the same subgroup and the hyperplane through these points is the corresponding hyperplane.*

Proof. The only thing we need to check is that the action of a p -Sylow subgroup does indeed have exactly p fixed points in $\mathbb{P}^{\frac{p-1}{2}}$, which amounts to proving that the matrix associated to a generator of the chosen subgroup has exactly $\frac{p-1}{2}$ different eigenvalues. Since all p -Sylow subgroups are conjugated, it suffices to prove this for the subgroup generated by

$$M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \in PSL_2(p).$$

Calculating the action of $e_1 e_2$ on an algebra corresponding to the point $(a_1, \dots, a_{\frac{p-1}{2}})$, we find that the new generators y_i are equal to

$$y_i = \omega^{-\frac{i(i+1)}{2}} x_i.$$

If we want to calculate the action of this element on a point $(a_1, \dots, a_{\frac{p-1}{2}})$,

we find that

$$\begin{aligned}
y_0 y_i + y_i y_0 &= \omega^{-\frac{i(i+1)}{2}} (x_0 x_i + x_i x_0) \\
&= \omega^{-\frac{i(i+1)}{2}} a_i x_{\frac{i}{2}}^2 \\
&= \omega^{-\frac{i(i+1)}{2}} a_i \omega^{\frac{i}{2}(\frac{i}{2}+1)} y_{\frac{i}{2}}^2 \\
&= \omega^{\frac{-i^2}{4}} a_i y_{\frac{i}{2}}^2.
\end{aligned}$$

This makes it clear that the corresponding matrix is a diagonal matrix.

Now, if $\omega^{\frac{-i^2}{4}} = \omega^{\frac{-j^2}{4}}$, then we must have $i^2 \equiv j^2 \pmod{p}$ or put differently, $i \equiv \pm j \pmod{p}$. Since we have $1 \leq i, j \leq \frac{p-1}{2}$, this ensures that $i = j$. \square

5.3 Character series

In the context of finding noncommutative algebras with properties similar to the polynomial rings, one of the similarities we would like to study are the character series. In this subsection we prove the following theorem.

Theorem 5.5. *If the algebra $\mathfrak{C}(a_0 : \dots : a_{\frac{p-1}{2}})$ satisfies the conditions of theorem 2.3, it has the same character series as the polynomial ring in p variables.*

First, we need some representation theory of H_p . More importantly, we need to find the decompositions of tensor products of simple p -dimensional representations. Let (V, φ) be the simple representation corresponding to $\varphi(z) = \omega I$ and let (V_i, φ_i) be the simple representation defined by $\varphi_i(z) = \omega^i I, i = 1, \dots, p-1$. Let $W_{i,j}, \chi_{i,j}$ be the 1-dimensional representations of H_p such that $\chi_{i,j}(e_1) = \omega^i, \chi_{i,j}(e_2) = \omega^j$ (in particular, $T = \chi_{0,0}$) and define $W = \oplus_{i,j=0}^{p-1} W_{i,j}$. Then character decomposition shows that

$$\begin{aligned}
V \otimes W_{i,j} &= V, \\
V \otimes V_i &= V_{i+1}^{\oplus p}, i \neq p-1, \\
V \otimes V_{p-1} &= W.
\end{aligned}$$

For example, the tensor algebra $T(V)$ has as character series

$$\begin{aligned}
Ch_{T(V)}(z^k, t) &= \frac{1}{1 - \omega^k p t}, \\
Ch_{T(V)}(e_1^k e_2^l, t) &= 1, (k, l) \neq (0, 0).
\end{aligned}$$

To find the character series of our graded Clifford algebras, we will use the fact that such an algebra is a free module of rank 2^p over a polynomial ring in p variables, with a basis found by the ordered monomials

$$\{x_{i_1} \dots x_{i_k} | 0 \leq i_1 < i_2 < \dots < i_k \leq p-1, 0 \leq k \leq p-1\}.$$

So in order to find the character series of the regular Clifford algebras, we first need to find the character series of $S(V) = \mathbb{C}[V]$. Since this is a Koszul algebra, it is more convenient to calculate the character series of $\wedge(V) = (S(V)!)^*$, for this is a finite dimensional algebra and then use equation 2.1. In the tensor algebra $T(V)$, every degree not divisible by p decomposes as one simple representation with a certain multiplicity. From this we immediately deduce

$$\begin{aligned} \wedge^0(V) &= T, \\ \wedge^i(V) &= V_i^{\oplus \frac{\binom{p}{i}}{p}}, 0 \neq i \neq p. \end{aligned}$$

In degree p , it is easily checked that $x_0 \wedge \dots \wedge x_{p-1}$ is fixed by H_p and thus $\wedge^p(V) = T$. Therefore, we have

$$\begin{aligned} Ch_{\wedge(V)}(z^k, t) &= (1 + \omega^k t)^p \\ Ch_{\wedge(V)}(e_1^k e_2^l, t) &= 1 + t^p, (k, l) \neq (0, 0). \end{aligned}$$

We can now use equation 2.1 to find the character series of $S(V)$.

Theorem 5.6. *The character series of $S(V)$ is given by*

$$\begin{aligned} Ch_{S(V)}(z^k, t) &= \frac{1}{(1 - \omega^k t)^p} \\ Ch_{S(V)}(e_1^k e_2^l, t) &= \frac{1}{1 - t^p}, (k, l) \neq (0, 0). \end{aligned}$$

If we want to calculate the character series of the regular Clifford algebra, we use the fact that this algebra is a free module of rank 2^p over the polynomial ring $\mathbb{C}[x_0^2, \dots, x_{p-1}^2]$. The character series of $\mathbb{C}[x_0^2, \dots, x_{p-1}^2]$ is given by

$$\begin{aligned} Ch_{\mathbb{C}[x_0^2, \dots, x_{p-1}^2]}(z^k, t) &= \frac{1}{(1 - \omega^{2k} t^2)^p} \\ Ch_{\mathbb{C}[x_0^2, \dots, x_{p-1}^2]}(e_1^k e_2^l, t) &= \frac{1}{1 - t^{2p}}, (k, l) \neq (0, 0). \end{aligned}$$

As an H_p -representation, the vector space with basis

$$\{x_{i_1} \dots x_{i_k} | 0 \leq i_1 < i_2 < \dots < i_k \leq p-1, 0 \leq k \leq p\}$$

is the same as $\wedge(V)$. Therefore, the character series of the graded Clifford algebra is given by

$$\begin{aligned} Ch_{\mathfrak{C}(a_0 : \dots : a_{\frac{p-1}{2}})}(z^k, t) &= Ch_{\mathbb{C}[x_0^2, \dots, x_{\frac{p-1}{2}}^2]}(z^k, t) Ch_{\wedge(V)}(z^k, t) \\ &= \frac{(1 + \omega^k t)^p}{(1 - \omega^{2k} t^2)^p} = \frac{1}{(1 - \omega^k t)^p}, \\ Ch_{\mathfrak{C}(a_0 : \dots : a_{\frac{p-1}{2}})}(e_1^k e_2^l, t) &= Ch_{\mathbb{C}[x_0^2, \dots, x_{\frac{p-1}{2}}^2]}(e_1^k e_2^l, t) Ch_{\wedge(V)}(e_1^k e_2^l, t) \\ &= \frac{1 + t^p}{1 - t^{2p}} = \frac{1}{1 - t^p}, (k, l) \neq (0, 0). \end{aligned}$$

This is indeed the character series of $S(V)$ as expected.

5.4 The character series of the center

We again assume that the requirements of theorem 2.3 are fulfilled. Since $p \geq 5$ is prime, it is not divisible by 2. This means that the center of the Clifford algebra $\mathfrak{C}(a_0 : \dots : a_{\frac{p-1}{2}}) = \mathfrak{C}(a_1, \dots, a_{\frac{p-1}{2}})$ is a quadratic extension of the polynomial ring $\mathbb{C}[x_0^2, \dots, x_{\frac{p-1}{2}}^2]$. The extra element is given by an element of degree p , $0 \neq g \in Z(\mathfrak{C}(a_1, \dots, a_{\frac{p-1}{2}}))_p$ and satisfies the equation $g^2 = \det(M)$, with M the quadratic form associated to $\mathfrak{C}(a_1, \dots, a_{\frac{p-1}{2}})$. Since H_p works as algebra automorphisms, it follows that the center is fixed by H_p and so $\mathbb{C}g$ is a 1-dimensional H_p -representation. We will now prove

Proposition 5.7. *As an H_p -representation, $\mathbb{C}g \cong T$.*

Proof. The proposition is equivalent with the statement that $\mathbb{C}\det(M) \cong T$, so we will prove this fact. Since $g^2 = \det(M)$, $\mathbb{C}\det(M)$ is indeed a H_p -representation. Fix a basis for $\mathfrak{C}(a_1, \dots, a_{\frac{p-1}{2}})_{2p}$ determined by it being a free module over $\mathbb{C}[x_0^2, \dots, x_{\frac{p-1}{2}}^2]$ of rank 2^p with the ordered monomials as basis and decompose this vector space as simple H_p -representations. Calculating the determinant of M , we get an element of the center of the form

$$\det(M) = (2^p + f(a_1, \dots, a_{\frac{p-1}{2}}))x_0^2 x_1^2 \dots x_{\frac{p-1}{2}}^2 + \dots$$

with $2^p + f(a_1, \dots, a_{\frac{p-1}{2}}) \neq 0$ a polynomial (not 0 because for the quantum plane we have $f(0, \dots, 0) = 0$). For the other elements of our basis, we get coefficients which are polynomials in $a_1, \dots, a_{\frac{p-1}{2}}$. Because

$2^p + f(a_1, \dots, a_{\frac{p-1}{2}}) \neq 0$, we have a Zariski-open subset of $\mathbb{A}^{\frac{p-1}{2}}$ for which $\mathbb{C}det(M)$ is indeed the trivial representation. Since $\mathbb{C}det(M)$ is always a H_p -representation, this means that on the same Zariski-open subset the coefficients of all the nontrivial representations in $\mathfrak{C}(a_1, \dots, a_{\frac{p-1}{2}})_{2p}$ are 0. Since these coefficients are also polynomial functions on $\mathbb{A}^{\frac{p-1}{2}}$, they must be 0 on $\mathbb{A}^{\frac{p-1}{2}}$. This means that $\mathbb{C}det(M) \cong T$. \square

From this, one easily finds that the character series of $Z(\mathfrak{C}(a_1, \dots, a_{\frac{p-1}{2}}))$ is given by

$$\begin{aligned} Ch_{Z(\mathfrak{C}(a_1, \dots, a_{\frac{p-1}{2}}))}(z^k, t) &= \frac{1 + t^p}{(1 - \omega^{2k} t^2)^p}, \\ Ch_{Z(\mathfrak{C}(a_1, \dots, a_{\frac{p-1}{2}}))}(e_1^k e_2^l, t) &= \frac{1 + t^p}{1 - t^{2p}} \\ &= \frac{1}{1 - t^p}, (k, l) \neq (0, 0). \end{aligned}$$

6 Future work

It is our hope to extend the dualities between the points necessary to make a compactification of $X(5)$ and the points isomorphic to the quantum space to the entire curve $\overline{X}(5)$. Hopefully there will be a subset of \mathbb{P}^2 that will determine 5-dimensional Sklyanin algebras depending on an elliptic curve and a point of order 2. The quadratic form associated to this algebra will be crucial in understanding the representation theory and the underlying H_5 -action, which will work on the variety that parametrizes representations of dimensions 1, 2 or 4. Further generalizations to dimension p , p being a prime number, will also be worked on in the future.

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